ON THE FORMULATION OF BOUNDARY CONDITIONS OF THE SIMPLIFIED EQUATIONS OF SHELLS OF REVOLUTION

PMM Vol. 34, №1, 1970, pp. 84-94
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(Received December 9, 1968)

The state of stress in a shell is frequently split up into two parts: one smooth part (the fundamental part of the solution), and one rapidly decaying part from the boundary into the interior of the region. (The usual edge effect [1] and the simple edge effect in vibrations [2] or stability [3].)

Below we present noncontradictory boundary conditions for the degenerate problem (the fundamental part of the solution). A theorem is formulated on the perturbation of the linear boundary value problem of the ordinary differential equation by an operator of higher order, whereby the small parameter enters not only the perturbed operator (see [4], supplement III), but also the boundary conditions (small parameter characterizes thinness of the shell). In essence, the formulation of Theorem 7 from [4] (supplement III) is made suitable to problems arising in the theory of shells [1]. The boundary conditions of the degenerate problem of the system of ordinary differential equations are discussed.

Examples are given of the simplified theories and associated boundary conditions for certain types of edge supports of the shells of revolution. As a special case the simplified theories for cylindrical shells are considered. A comparison for the boundary conditions of simplified equations with results by other authors are given.

1. Let us consider an ordinary differential equation with constant coefficients

$$A_\varepsilon u = \varepsilon^l a_{k+l} \frac{d^{k+l}u}{dx^{k+l}} + \ldots + a_k \frac{d^k u}{dx^k} + \ldots + a_0 u = 0 \qquad (0 \leqslant x \leqslant 1) \tag{1.1}$$

Let Eq. (1.1) with boundary conditions containing small parameters

$$L_{im}u = \sum_{j=0}^{k+l-1} b_{ijm} \frac{d^j u}{dx^j} = \varphi_{im} \qquad \begin{matrix} (x=0, & m=0; & i=1,\ldots,r) \\ (x=1, & m=1; & i=1,\ldots, & k+l-r) \end{matrix} \tag{1.2}$$

have a unique solution for arbitrary right-hand sides in (1.2), where ε is a sufficiently small positive quantity.

Let us assume that the characteristic equation, corresponding to (1.1), has l large roots of order $O(\varepsilon^{-1})$ and k finite roots, whereby among the l large roots p have a negative real part, while $(l-p)$ possess a positive real part.

Let us denote the boundary conditions written out in canonical form if they are solved for terms characterizing the maximum of the parameter μ ($\mu = n - k^*$, k^* are integers).

$$L_{im}{}^* u = \varepsilon^{k^*} \frac{d^n u}{dx^n} + \sum_{j=0}^{n_i^*-1} c_{ijm} \frac{d^j u}{dx^j} = \Phi_{im} \tag{1.3}$$

$$x=0, \quad m=0; \quad i=1,\ldots,r; \quad \mu_1{}^0 < \mu_2{}^0 < \ldots < \mu_r{}^0$$

$$x=1, \quad m=1; \quad i=1,\ldots, \quad k+l-r; \quad \mu_1{}^* < \mu_2{}^* < \ldots < \mu_{k+l-r}^*$$

Here μ, n, k^* are functions with integer arguments

$$\mu = \mu\ (i,\ m), \quad n = n\ (i,\ m), \quad k^* = k\ (i,\ m)$$

Let us note that in passing from the ith boundary condition to the $(i+1)$-th one, the parameter μ is increased, i.e. $\mu\ (i) < \mu\ (i+1)$. In relations (1.3) there are no small parameters ε with negative powers, i.e. $k\ (i,\ m) \geqslant 0$ for all i and m. Furthermore, the small parameter ε is not a common factor in the expression $L_{im}{}^* u = \Phi_{im}$. In the special case, when $k\ (i,\ m) \equiv 0$, the formulation of the boundary conditions (1.3) coincides with the canonical form introduced in supplement III of [4].

Theorem 1. If the problem,

$$A_0 u_0 = a_k \frac{d^k u_0}{dx^k} + \ldots + a_0 u_0 = 0 \tag{1.4}$$

$$L_{im}{}^0 u_0 = \Phi_{im}{}^0 \ (x = 0,\ \ m = 0;\ i = 1, \ldots,\ r - p;\ x = 1,\ \ m = 1;\ \ i = 1, \ldots,\ k - r + p) \tag{1.5}$$

has a unique solution, the solution u_ε of the problem (1.1), (1.2) approaches the solution of the problem (1.4), (1.5) if $\varepsilon \to 0$.

It is necessary to note that $L_{im}{}^\circ$, $\Phi_{im}{}^\circ$ in (1.5) differ from (1.3) because $L_{im}{}^\circ$, $\Phi_{im}{}^\circ$ do not contain any terms with small parameter ε.

Proof. The asymptotic solution of problem (1.1), (1.2) is sought in the form

$$u_\varepsilon = \sum_{s=0}^{S} \varepsilon^s u_s + \varepsilon^\alpha \sum_{s=0}^{S} \varepsilon^s v_s + \varepsilon^\beta \sum_{s=0}^{S} \varepsilon^s w_s + z_S \tag{1.6}$$

Here z_S is the remainder in the expansion; v_s and w_s are functions of boundary layer type in the vicinity of points $x = 0$ and 1, respectively; both these functions are constructed taking into account the smoothing factor, as was done in [4], while α and β are some constants.

The asymptotic solution (1.6) is substituted into (1.1) and (1.3), whereby for v_s and w_s the representation of the operators A_ε and $L_{im}{}^*$ is used in which the scale in the vicinity of the points $x = 0$ and $x = 1$ is taken into account, just as in [4]. In the expressions obtained the coefficients of like powers in ε are set equal to each other [4]. The fundamental part of the solution is determined from $A_0 u_0 = 0$. The boundary conditions for this equation are (1.5).

From boundary conditions for solutions of boundary layer type in the vicinity of $x = 0$ and 1 the constants α and β are determined. For example, for a correct construction of the iteration process α is determined from the relation, see (1.3),

$$\alpha = n\ (r - p + 1,\ 0) - k\ (r - p + 1,\ 0)$$

The subsequent approximations are problems obtained after setting equal to zero expressions with the subsequent powers of ε. One should not forget to take into account that for u_s, v_s $(s \neq 0)$ the boundary conditions are somewhat more complicated as compared to the corresponding boundary conditions in [4] (supplement III). This is due to the fact that in addition to the boundary layer, which is determined by the perturbed operator of higher order, there are present perturbations of boundary conditions both in the fundamental part of the boundary conditions

$$L_{im}{}^* u_s = \Phi_{im} \qquad (m = 0,\ i = 1, \ldots,\ r - p;\ m = 1,\ i = 1, \ldots,\ k - r + p)$$

and in the boundary conditions for the solutions of boundary layer type. The perturbation of the boundary condition is discussed in [4] (Sect. 4, item 2).

It is now obvious that the scheme for proving Theorem 7 from [4] (supplement III) can be completely applied to the present theorem.

Consequently, the asymptotic representation of solution (1.1) and (1.2) in the form of (1.6) is valid, and therefore $\lim u_\varepsilon = u_0$ for $\varepsilon \to 0$. This is what we had to prove.

Thus, the theorem permits to obtain from (1.2) noncontradictory boundary conditions (1.5) for the degenerate (simplified) equation $A_0 u_0 = 0$. Indeed, the solution u_ε of the complete problem (1.1), (1.2) differs from the solution of the degenerate problem (1.4), (1.5) only through terms with small parameter ε^s, whereby $s > 0$.

If, by some means, simplified boundary conditions are obtained from (1.2) for the same equation (1.4), different from (1.5), then the solution in the form (1.6) cannot be constructed, since the expansion (1.6) is unique. And consequently, such simplified boundary conditions are contradictory, because the solution u_ε of the complete problem and the solution u_0^* of the degenerate problem have nothing in common in this case as $\varepsilon \to 0$.

Example 1. Let us consider a differential equation of second order, which is perturbed by an operator of fourth order,

$$\varepsilon^2 u^{(4)} - u'' = 0$$

for boundary conditions

$$u_0 + u'' = 2, \qquad u' - \varepsilon^k u'' = 1 \ (x = 0,1)$$

The conditions of the theorem are fulfilled and, consequently, the boundary conditions for the degenerate equation $u^{\mathrm{II}} = 0$ ($\varepsilon \to 0$) depending upon the value of the parameter k take on the form

$$\begin{array}{cccc} k = 0 & k = 1 & k \geqslant 2 & \\ u_0 + u_0'' = 2 & u_0 + u_0'' - u_0' = 1 & u_0' = 1 & (x = 0,\ 1) \end{array}$$

We note that for $k = 0$ we may use Theorem 7 from [4] (supplement III), and for $k > 0$ it is necessary to apply the formulated Theorem 1.

2. Problems of the linear theory of shells are problems on a system of two differential equations for functions of normal deflection and the stresses [1]. Generally speaking, such a system cannot be reduced to a single equation, and therefore we shall adapt the formulation of Theorem 1 to the investigation of systems of ordinary differential equations with analytical coefficients, when the small parameter ε (in the theory of shells it characterizes the thinness of the structure) enters the system of the equations and the boundary conditions. Let us make use of the vector notations

$$B_\varepsilon U' = AU \qquad (0 \leqslant x \leqslant 1) \tag{2.1}$$

$$L_m U = \varphi_m \qquad (m = 0 \quad \text{for} \quad x = 0, \quad m = 1 \quad \text{for} \quad x = 1) \tag{2.2}$$

Here A and B_ε are quadratic matrices with analytical coefficients of order $l + k$, whereby in the matrix B_ε only the terms b_{ij} on the main diagonal $i = j$ are different from zero; further, $b_{ii} = O(\varepsilon)$ for $i > k$, and $b_{ii} = 0\,(1)$ for $i \leqslant k$; U is a vector function; L_m are matrices (for $m = 0$ the number of rows is r, for $m = 1$ the number of rows is $k + l - r$), φ_m is a vector. The matrices B_ε, A and L_m contain terms with small parameters. Regarding the possibility of reducing an arbitrary system to the form (2.1) see [5], Sects. 37, 38.

We consider such problem (2.1) in which for $\varepsilon = 0$ the order of the degenerate (reduced according to [5]) system is $k < k + l$. Let us assume that the characteristic polynomial of order $k + l$ (see Sect. 1), corresponding to $A - \lambda B_\varepsilon$ possesses for each $x \in [0, 1]$ l large roots of order $O(\varepsilon^{-1})$ and k finite roots, whereby among the l

large roots p possess a negative real part, while $l - p$ possess a positive real part.

Note. In the present analysis the case is not studied when the number of finite roots and roots of order $O(\varepsilon^{-1})$ is changing as $x \in [0,1]$, see, for instance, [6].

In analogy to Sect. 1 we introduce the definition: the boundary conditions are called as being written in canonical form if they are solved with respect to terms which characterize the maximum of the parameter μ and in the passage from a previous to a subsequent boundary condition the parameter μ increases $(\mu(i) < \mu(i+1))$

$$L_m{}^*U = \Phi_m \qquad (m = 0 \quad \text{for} \quad x = 0, \quad m = 1 \quad \text{for} \quad x = 1) \tag{2.3}$$

The rectangular matrices $L_m{}^*$ are of the form

$$L_m{}^* = \begin{Vmatrix} L_m{}^0 & G_\sigma \\ C_5 & 'C_7 \end{Vmatrix} \tag{2.4}$$

The number of columns in $L_m{}^\circ$ coincides for $m = 0, 1$ and is equal to k, while the number of rows is $r - p$ for $m = 0$ and $k - r + p$ for $m = 1$. Just as in Sect. 1, no boundary conditions (2.3) contain ε with a negative power and it is not a common factor. The parameter μ is calculated in accordance with the following rule:

$$\mu = -k^* - n \qquad (k^* \geqslant 0, \quad n \geqslant 0)$$

where $k^* = k(i, j)$, $n = n(i, j)$ are powers of the small parameter in terms with $L_m{}^*$ and in the solution of the boundary layer type v_i near $x = 0$ and w_i near $x = 1$ $(c_{0j} = 0(1),\ c_{1j} = 0(1))$

$$\begin{aligned} v_i &= \sum_{j=1}^{p} c_{0j}\varepsilon^n \exp\left(-\frac{q_{0j}x}{\varepsilon}\right) \qquad (i = k+1, \ldots, k+p) \\ w_i &= \sum_{j=1}^{l-p} c_{1j}\varepsilon^n \exp\frac{q_{1j}(x-1)}{\varepsilon} \qquad (i = k+p+1, \ldots, k+l) \end{aligned} \tag{2.5}$$

The boundary layers v_i and w_i are decaying from the boundary into the interior of the region of the solution of the systems of equations with constant coefficients

$$B^{*\circ}V' = A^{*\circ}V, \qquad B^{*1}W' = A^{*1}W$$

The quadratic matrices B^{*0}, B^{*1}, A^{*0}, A^{*1} having the order l, correspond to the right lower corners of matrices B_ε and A, if we set in the latter $x = 0$ and $x = 1$,

$$B_\varepsilon = \begin{Vmatrix} B_0 & C_2 \\ C_1 & B^* \end{Vmatrix}, \qquad A = \begin{Vmatrix} A_0 & C_4 \\ C_3 & A^* \end{Vmatrix} \tag{2.6}$$

Let the problem (2.1), (2.2) for an arbitrary vector φ_m have a unique solution if $\varepsilon > 0$ is a sufficiently small quantity, and the number of boundary conditions (2.2) for $x = 0$ and 1 is larger than the corresponding number p and $l - p$ $(r > p,\ k + l - r > l - p)$, i.e. the condition of regularity of the degeneration [4] is fulfilled. "Roughly speaking, the condition of regularity indicates that the boundary conditions should not be distributed too unevenly between two boundary points" ([5], p. 272).

Theorem 2. If the problem

$$B_0U_0' = A_0U_0 \tag{2.7}$$

$$L_m{}^0U_0 = \Phi_m \ (m = 0 \quad \text{for} \quad x = 0, \ m = 1 \quad \text{for} \quad x = 1) \tag{2.8}$$

possesses a unique solution, then the solution U_ε of the problem (2.1),(2.2) approaches the solution of the problem (2.7),(2.8), as $\varepsilon \to 0$.

The matrices $B_0, A_0, L_m{}^0$ and the vector $\Phi_m{}^0$ are determined above in (2.3),(2.4) and (2.6).

The proof of Theorem 2 is analogous to the proof of Theorem 7 in [4] (supplement III), see discussion in Sect. 1.

Thus, the theorem formulated permits to obtain noncontradictory boundary conditions (2.8) for the degenerate problem $B_0U_0' = A_0U_0$.

Example 2. We consider the system of differential equations of fourth order

$$B_\varepsilon U' = AU \qquad (0 \leqslant x \leqslant 1)$$

$$B_\varepsilon = \begin{Vmatrix} b_{11} & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & \varepsilon b_{33} & 0 \\ 0 & 0 & 0 & -\varepsilon b_{44} \end{Vmatrix}, \qquad A = \begin{Vmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{Vmatrix}, \qquad U = \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Vmatrix}$$

with the boundary conditions

$$u_1 + u_3 = 2, \qquad u_2 + \varepsilon^{k-1}u_4 = 1 \qquad (x = 0,1)$$

Here b_{ij} and a_{ij} are analytical functions, and k is some number. If the conditions of Theorem 2 are satisfied, then for the degenerate (simplified) system of second order ($\varepsilon \to 0$)

$$b_{11}u_{01}' = a_{12}u_{02}, \qquad b_{22}u_{02}' = 0$$

the noncontradictory boundary conditions are

$k = 0$	$k = 1$	$k \geqslant 2$	
$u_{01} + u_{03} = 2$	$u_{01} + u_{03} - u_{02} = 1$	$u_{02} = 1$	$(x = 0.1)$

We note that for $b_{11} = b_{22} = b_{33} = b_{44} = 1$ and $a_{12} = a_{22} = a_{34} = a_{43} = 1$ this example coincides with Example 1.

3. Let us consider only shells of revolution. Problems of the theory of shells are problems on a system of two differential equations with a small parameter of high derivatives [1]

$$h_0^2N\Phi + L\Phi + \lambda M\Phi = F \tag{3.1}$$

The boundary conditions for system (3.1) are of the form

$$R_i\Phi = \varphi_i \qquad (i = 1, 2, 3, 4) \tag{3.2}$$

Here L, N, M are linear differential operators with two independent variables, whereby the order of operator N is larger than that of L, while M has either the same order as L or larger; λ is a parameter of the loading or of the natural frequency (for strength problems $\lambda \equiv 0$); R_i are linear differential operators, determined on the contour of the shell; φ_i and F are specified loadings (displacements) along the contour and on the middle surface, respectively; h_0 is a dimensionless thickness of the shell.

Generally speaking, the small parameter h_0 may not enter into the operators R_i. For example, for the theory of shallow shells the boundary conditions are written down without a small parameter. Yet the more refined versions of the theory of shells contain a small parameter h_0 multiplying both the smallest and the largest derivatives in R_i.

Furthermore, we have to take into account that F and φ_i may be oscillating functions. Therefore it makes sense to consider "problems in which for an unbounded decrease of

h_0 in Eq. (3.1) the oscillatory behavior of the boundary conditions (3.2) or of the free term in (3.1) is increased according to a definite law", see [1], introduction.

And finally, for problems on free vibrations and stability "the smallest eigenvalue λ in several cases corresponds to eigenfunctions with a rather large... number of nodal lines" [1].

In all these three cases we obtain after separation of variables a system of ordinary differential equations with some boundary conditions. Both in the system and in the boundary conditions terms with a small parameter are present (the separation of variables may be always achieved for complete shells of revolution if the ends of the shells are normal to the axis).

The obtained problem for the system of equations can often not be reduced to problems (1.1), (1.2); (2.1), (2.2). This is connected with the fact that the order of large roots of the characteristic equation may differ from $O(\varepsilon^{-1})$. Therefore, instead of ε we introduce into the system and the boundary conditions another small parameter $\delta = \varepsilon^{\gamma}$, where γ is the order of the large roots with respect to the characteristic equation in ε, and therefore either Theorem 1 or 2 may be applied, i.e. the last p and $l-p$ boundary conditions un (1.3) or in (2.3) are omitted as well as the terms in small parameters.

In the case when the order of the large roots is different, for example $O(\varepsilon^{-\gamma_1})$ and $O(\varepsilon^{-\gamma_2})$, among the two the larger number $\gamma_1 > \gamma_2$ is selected. All arguments are carried out first for the roots of order $-\gamma_1$ $(\delta_1 = \varepsilon^{\gamma_1})$. We obtain the first degenerate (reduced) problem (see Theorems 1, 2). This first degenerate problem is simplified further. The simplifications are now carried out for $\delta_2 = \varepsilon^{\gamma_2}$. Applying once more either Theorem 1 or 2, we obtain the second degenerate (simplified) problem.

Thus, for the simplified theories of shells a procedure is indicated to obtain noncontradictory boundary conditions (1.5) or (2.7), whereby the solution of the complete equations of the theory of shells does not differ considerably from the solution of the simplified equations if the parameter δ is small.

4. Let us consider some examples in greater detail. Let us estimate the influence of the type of support at the ends of the cylindrical shell, subjected to external transverse or hydrostatic pressure on the parameter of critical loading, if the analysis is carried out using the approximate theory (theory of shallow shells) and a refined theory of type [7]. The estimates obtained will be compared with numerical calculations from [3, 8].

In the problem considered, to the smallest eigenvalue λ there corresponds a definite number of waves of loss of stability n along the circumference. If the support is a hinge or a clamping, then

$$n^2 = O(\varepsilon^{-1}), \qquad \varepsilon = (h / R)^{1/2}$$

Here R is the radius of the shell and h its thickness.

We investigate the characteristic equation of the differential expression, which is obtained from (3.1) after separation of variables. The order of the roots of the edge effect, to which there corresponds an oscillating solution, is $-O(\varepsilon^{-1})$. The roots which generate a nonoscillatory edge effect, have the order $O(\varepsilon^{-1/2})$. The fundamental part of the solution, which embraces the whole region, is oscillating [3, 8].

The boundary conditions of the hinged or clamped type considered in [3, 8], after separation of variables contain small parameters. Exceptions are the boundary conditions

$$N_1 = v = w = M_1 = 0 \tag{4.1}$$

Here N_1 is the force along the generator, w is the nominal deflection, v is the displacement along the circumferential coordinate, M_1 is the bending moment. These conditions take on the form after the separation of variables

$$f = f'' = f^{(4)} = f^{(6)} = 0 \qquad (4.2)$$

If in a complete equation of eighth order with boundary conditions (4.1) at both ends the small parameter is approaching zero $\varepsilon \to 0$ $(h \to 0)$, then the boundary conditions for the degenerate (simplified) problem (see [4], supplement III) are $f_0 = f_0'' = 0$ (the degenerate equation is of fourth order), while the boundary conditions for the still more simplified theory (equation of second order) are $f_0 = 0$.

For other types of support the canonical form by contrast to (4.2) is determined by the maximum of parameter μ (see Sect. 1). The canonical form of the boundary conditions for one of support versions (the degenerate equation is of fourth order)

$$N_1 = v = w = w_{,x} = 0 \qquad (4.3)$$

$$f = 0\ (\mu_1 = 0), \quad f'' = 0\ (\mu_2 = 2), \ \nabla\nabla \ f' = 0\ (\mu_3 = 3), \ f^{(4)} = 0\ (\mu_4 = 4)$$

where x is a coordinate along the cylinder axis, $\nabla f = f'' - n^2 f$. The canonical form of the boundary conditions for clamping

$$N_1 = S = w = M_1 = 0 \qquad (4.4)$$

is more unwieldy than (4.3). Formally Eqs. (4.4) may be rewritten as

$$f'' = f''' = 0 \qquad (4.5)$$

$$\varepsilon^2 f^{(4)} \sum_{j=0}^{\infty} a_j \varepsilon^j + f = 0, \quad \varepsilon^2 \left(\varepsilon f^{(6)} \sum_{j=0}^{\infty} b_j \varepsilon^j + f^{(4)} \sum_{j=0}^{\infty} c_j \varepsilon^j \right) + f = 0$$

The series are assumed to be convergent. If in the expression for M_1 Poisson's ratio is set equal to zero, then before the parenthesis in the last condition (4.5) the small parameter enters in its first power. It is precisely this case which is being analyzed in the sequel.

Following the rule, the boundary conditions should be given the form (the degenerate equation is of fourth order)

$$a_0 \left(\varepsilon^2 f^{(4)} - f''\right) + \varepsilon^2 f^{(4)} \sum_{j=1}^{\infty} a_j \varepsilon^j + f = 0 \qquad (\mu_1 = 1)$$

$$f'' = 0\ (\mu_2 = 2), \qquad f''' = 0 \quad (\mu_3 = 3) \qquad (4.6)$$

$$\varepsilon \left(\varepsilon f^{(6)} \sum_{j=0}^{\infty} b_j \varepsilon^j + f^{(4)} \sum_{j=0}^{\infty} c_j \varepsilon^j \right) + f = 0 \qquad (\mu_4 = 4)$$

From the boundary conditions (4.3) and (4.6) in canonical form a judgement can be made concerning the order of corrections to the parameter of the critical loading λ_0. The parameter λ_0 is determined for a shell hinged at both ends (see (4.1)). For boundary conditions (4.3) and (4.6), as compared to (4.2), it is the third condition which appears to be "spoiled". Therefore the relative correction $|\lambda / \lambda_0|$ for conditions (4.3) or (4.6) cannot be large, as calculations have shown [3].

The largest relative correction is obtained if the fundamental boundary condition either the first or the second in (4.2) is "spoiled". This takes place when such support conditions of the shell are considered that the displacement u in the axial direction of the cylinder is constrained. For example, for the supports

$$u = w = S = w_{,x} = 0$$

the canonical form of the boundary conditions (the degenerate problem is of fourth order) is

$$f' = 0\ (\mu_1 = 1), \quad \nabla\nabla f = 0\ (\mu_2 = 2), \quad f''' = 0\ (\mu_3 = 3), \quad f^{(5)} = 0\ (\mu_4 = 5)$$

The calculations carried out in [8] have shown that the boundary condition $u = 0$ essentially changes the value of the critical load.

When shells with free edges are investigated, it is necessary to employ refined equations of stability and corresponding boundary conditions [7], for example [3], Sect. 2. The boundary conditions of the refined theories contain terms with small multipliers. At a free edge all force factors are equal to zero (Q_1^* is the generalized transverse force)

$$N_1 = S = M_1 = Q_1^* = 0 \tag{4.7}$$

A detailed writing out of (4.7) after separation of variables is given in Sect. 2 of [3]. The boundary conditions for the degenerate problem are formulated in dependence on the number n of circumferential waves. When $n = 2$ ($R / h = 100$), the boundary conditions of the degenerate problem of fourth order are of the form

$$f_0'' = 0\ (\mu_1 = 2),\ f_0''' = 0\ (\mu_2 = 3) \tag{4.8}$$

The degenerate problem of second order for the first form along the axial coordinate is without meaning since the nonoscillatory edge effect embraces the whole shell (see [3]).

In the case when $n^2 = O(\varepsilon^{-1})$ (this is the n for the smallest critical load of the second form of loss of stability along the axis of a coordinate of a cylindrical shell, one edge of which is hinged while the other is free, $R / h = 100$), the boundary conditions of the same limiting problem of fourth order takes on the form

$$f_0 - b_0 f_0''' = 0,\ f_0'' = 0 \tag{4.9}$$

The first condition (4.9) corresponds to the boundary condition, see (4.4) and (4.5) (Poisson's ratio is different from zero in M_1)

$$b_0(\varepsilon^3 f^{(6)} - f''') + b_1\varepsilon^4 f^{(6)} + \varepsilon^2 c_0 f^{(4)} + \varepsilon^3 f^{(6)} \sum_{j=2}^{\infty} b_j\varepsilon^j + \varepsilon^2 f^{(4)} \sum_{j=1}^{\infty} c_j \varepsilon^j + f = 0 \tag{4.10}$$

The passage from (4.10) to the first condition (4.9) is realized as $\varepsilon \to 0$. In condition (4.10) the terms are already omitted which have no influence on the result. These terms appear because more accurate expressions than (4.5) should be considered.

Since the root corresponding to the nonoscillatory edge effect for the second form, is of the order $O(\varepsilon^{-1/2})$, the boundary conditions of the degenerate problem of second order are $f_0'' = 0$. Evidently, the fundamental part of the second form of buckling along the axial coordinate is the half sine wave for the shell, whose one edge is hinged (4.1) and whose other end is free (4.7), see [3], Fig. 1 $m = 1$.

Thus, for the same equation of fourth order different boundary conditions (4.8), (4.9) are obtained, even though the original conditions of support coincided and have the form of (4.7).

The procedure of determining the boundary conditions for approximate theories of shells in [9, 10] does not always lead to results which coincide with those presented above.

The formulated Theorem 1 (we recall that this theorem is a different version of Theorem 7 from [4]) permits to approach systematically the formulation of the boundary conditions of the simplified equations of the theory of shells. Theorem 7 from [4] among the discussed five versions of boundary conditions is applicable directly only to (4.1), see (4.2).

5. The determination of noncontradictory boundary conditions requires in the general case for the shells of revolution rather cumbersome yet simple calculations.

Below we compare the noncontradictory boundary conditions of the degenerate problem of fourth order (see Theorem 2) and the boundary conditions for the simplified equations of stability of circular conical shells under hydrostatic pressure [6, 11]. The comparison is carried out for the end $x = 1$, (see [6]). All notation in this section coincides with the notation of paper [6], because it was suitably introduced there

$k^2 = O(\varepsilon^{-1/2})$	$\psi = \varphi = 0,$	$\psi = \varphi = 0$	$\psi = \varphi = 0$
$k^2 = O(\varepsilon^{-1})$	$\psi = \varphi = 0$	$\psi = p\psi' - \nu\varphi = 0$	$\psi = \varphi = 0$
$k^2 = O(\varepsilon^{-3/2})$	$\psi = \varphi = 0$	$\psi = \psi' = 0$	—

Here ψ and φ are functions of normal deflection and stresses along the generator, k is the circular frequency of change of the state of stress after loss of stability when the shell is passed along the parallel, ε is a small parameter. Column 1 corresponds to complete boundary conditions (1.5) from [6]; 2 to (1.4) from [6]; 3 to (10) from [11]. All boundary conditions of the first column coincide with the simplified boundary conditions (2.3) of system (2.1) [6], in the second column the last row coincides with the simplified conditions (2.2) of system (2.1) [6]. The line in the third column replaces a cumbersome expression. None of the versions of the boundary conditions of the last column coincides with the simplified boundary conditions (12) from [11].

The sequence of calculations in the determination of the noncontradictory boundary conditions of the degenerate problem given in the table is as follows: first we investigate the transformed systems of the equations which describe the stability of shells: (1.2) from [6] and (9) from [11]; we determine the four largest, by modulus, roots, whereby two of them possess a negative real part, while two have a positive real part; we introduce a corresponding small parameter (see Sect. 3); we determine the edge effects (2.5) and the degenerate system of the equations of fourth order (see (2.1) [6]); the boundary conditions (1.4) and (1.5) from [6] and (10) from [11] are written down in canonical form (2.3), (2.4) (the parameter μ is calculated first); now we have only to strike out the two last conditions in (2.3), in the remaining first two, if there are terms with the small parameter ε, the small parameter has to be made to approach zero $\varepsilon \to 0$; after all this we revert to the old notation.

Thus, as in the preceding section, the essential dependence of the boundary conditions of the degenerate problem on the frequency of oscillations k is uncovered. The existence of this phenomenon was indicated in [1], p. 4.

The effected separation of the state of stress of the shell of revolution into two parts permits the application in each part of special methods of solution, namely: the fundamental part of the solution (the degenerate problem with its own boundary conditions) is constructed suitably by variational or by numerical methods, because this part of the solution, as a rule, changes smoothly, and this solution is to be corrected by the edge effect, whose effective construction is well known [1, 4]. For example, in parers [6, 12] the fundamental part of the solution in stability problems of shells of revolution is written out in explicit form with great accuracy. Separation of the boundary conditions in the linear theory of shells permits the passage to the analysis of more complicated problems [13, 14], which arise in the analysis of thin-walled stiffened structures.

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